Space-Time Diagrams: Visualizing Special Relativity

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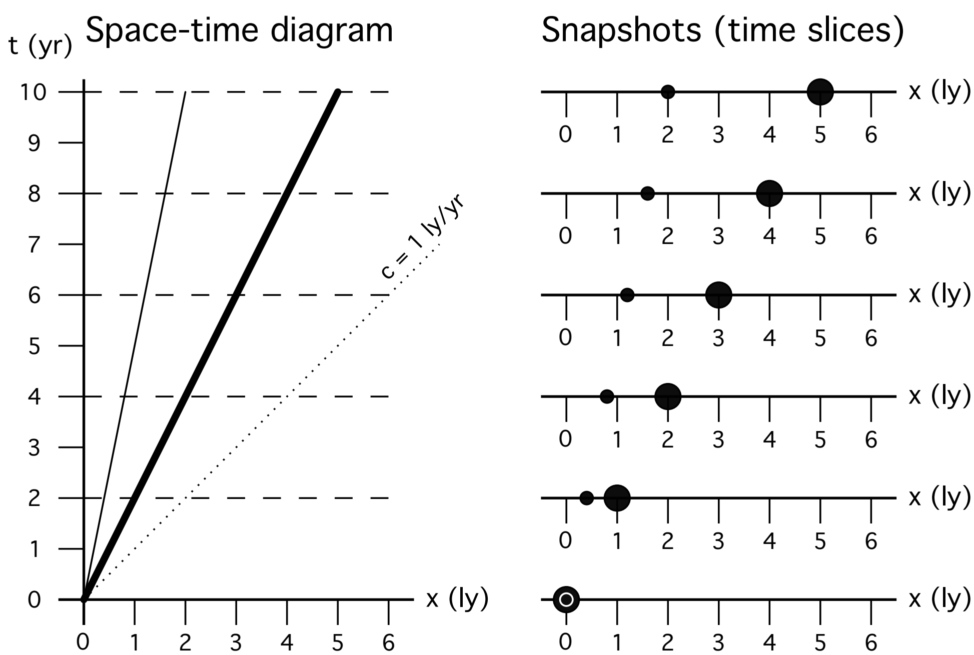
A space-time diagram shows the history of objects moving through space (usually in just one dimension). A specific point on a space-time diagram is called an "**event**."

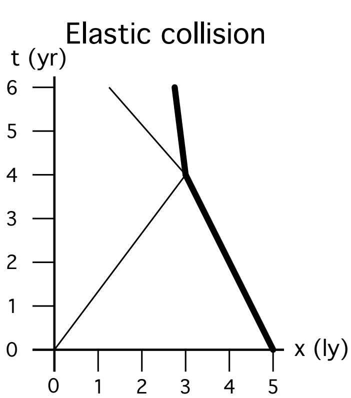
- To make a space-time diagram, take many snapshots of the objects over time and set them on top of each other. Lines in the diagram are like "contrails" through time.
- Given a diagram, to take a snapshot find the positions of each object on the appropriate "time slice" (the dashed lines below). **A time slice labels a set of "events" that the observer considers simultaneous.** We will draw time slices at equally spaced intervals so that each slice corresponds to one "tick" of the observer's clock.

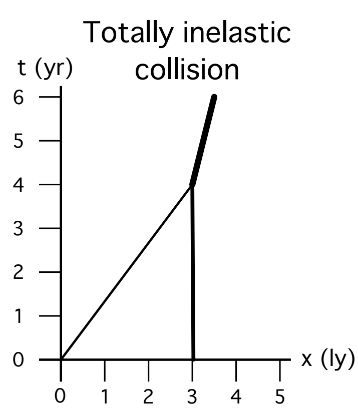


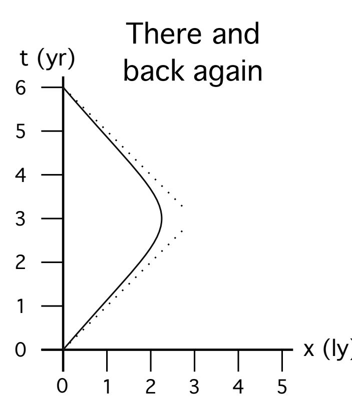
In these diagrams:

- The closer an object's path is to vertical, the slower it is moving: **slope = $1/v$**.
- A 45° angle corresponds to the speed of light: we will always measure time in years (yr) and space in lightyears (ly) (or in seconds and light-seconds, etc.). **Thus, $c = 1$ ly/yr.**

A few examples:







Thus far, this is just another way to visualize motion: it applies just as well to classical physics as it does to relativity. Relativity adds two new ingredients:

- Nothing can travel at more than a 45° angle from vertical (even for a moment): $\boldsymbol{v < c}$.
- **Different observers see different time slices!**

In particular, the time slices seen by the observer drawing the diagram (the observer "at rest") are the same horizontal lines as before. But the time slices seen by an observer traveling at speed v relative to the diagram's reference frame have the following properties (recall that $\gamma = {}^1\!/\!\sqrt{1-\frac{v^2}{c^2}}$, so for example, $v = \frac{1}{2}c$ gives $\gamma = 1.15$):

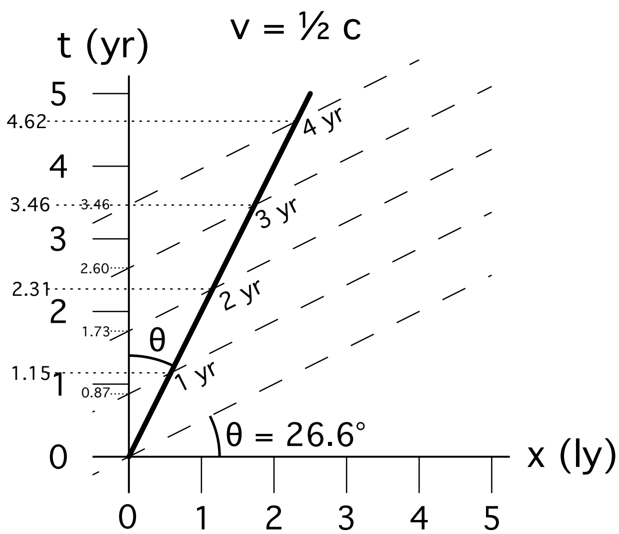
- The time slices **tilt up toward 45°** as v increases: they have **slope** $\boldsymbol{v/c^2} = v \cdot \frac{1\,\text{yr}^2}{\text{ly}^2}$.

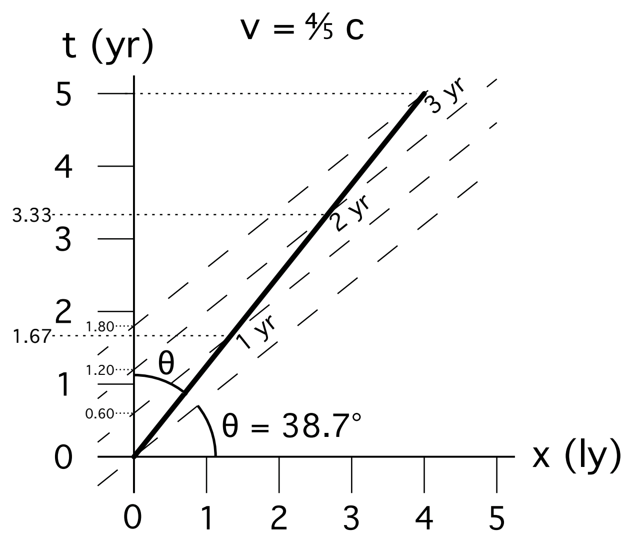
  As long as we draw c at 45°, this means that **a moving observer's time slices tilt up by the same angle $\boldsymbol{\theta}$ that her path tilts down** in our diagram. (*Technically, we are graphing the "Lorentz transformation" equations, plugging in various values for t'.*)

- Time slices get **squeezed together** as v increases. (Compare $v = \frac{1}{2}c$ and $\frac{4}{5}c$ below.)

  A moving observer's time slices spaced $\Delta t'$ apart are separated vertically on the diagram by $\boldsymbol{\frac{\Delta t'}{\gamma}}$ (e.g. for $v = \frac{1}{2}c$, 1 yr time slices intersect the t-axis every 0.87 yr).

- **A moving observer's "clock ticks" occur when her time slices intersect her path** on the diagram. Thus, an observer "at rest" sees the moving clock ticking *less* often: the *intersections* have larger vertical spacing, every $\boldsymbol{\Delta t = \gamma \Delta t'}$ (so the moving observer's $\Delta t' = 1\,\text{yr}$ time slice intersects her path at diagram time $\Delta t = 1.15\,\text{yr}$).





Example: Above, the diagram observer clearly sees moving clocks running slow: for each "1 yr" intersection along the moving path, his dotted horizontal time slices come more than 1 yr apart. For example, in the $v = \frac{1}{2}c$ diagram the moving observer's 2 yr "clock tick" is delayed until $t = 2.31\,\text{yr}$ (and when $v = \frac{4}{5}c$, it is delayed even more: until $t = 3.33\,\text{yr}$).

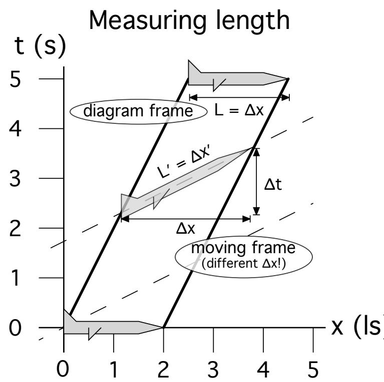
But the moving observer would say the same thing about the diagram observer! In the $v = \frac{1}{2}c$ diagram, the diagram observer's "clock tick" at $x = 0$ and $t = 2\,\text{yr}$ falls between the moving observer's 2 yr and 3 yr time slices. (In fact, it is at exactly $\Delta t' = 2.31\,\text{yr}$.) Both observers see the other clock running slow by the same amount. The page about the "twin 'paradox'" will give you some idea of why this isn't a contradiction.

In relativity (including space-time diagrams), time and space are mixed together in two **Not-Quite-Pythagorean Theorems for Space-Time** (a.k.a. the "space-time interval"):

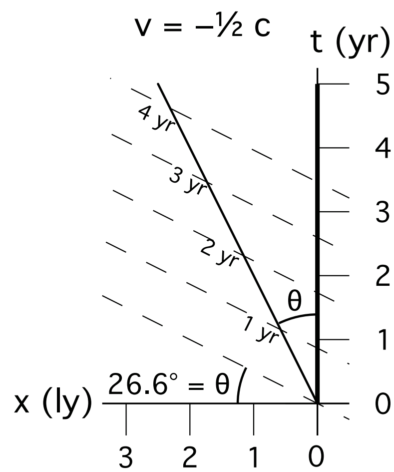
- The time $\Delta t'$ experienced along an observer's path obeys $(c\Delta t')^2 = (c\Delta t)^2 - (\Delta x)^2$.
- A length $\Delta x'$ measured along an observer's time slice obeys $(\Delta x')^2 = (\Delta x)^2 - (c\Delta t)^2$.

These are equivalent to $\Delta t = \gamma \Delta t'$ and $L = \frac{L'}{\gamma}$ *(just plug in $\Delta x = v\Delta t$, etc.)*. **WARNING! Never measure lengths at an angle on a spacetime diagram with a ruler**: lines on paper obey the *ordinary* Pythagorean theorem, not these special space-time rules.

To find **the length of an object**, an observer must **measure between points on the SAME time slice**, as shown. (Why? At right, if you located the tail at $x = 0$ when $t = 0$ and then located the nose at $x = 4.5$ ls ("light-seconds") when $t = 5$ s, you shouldn't conclude that the ship is 4.5 ls long!)



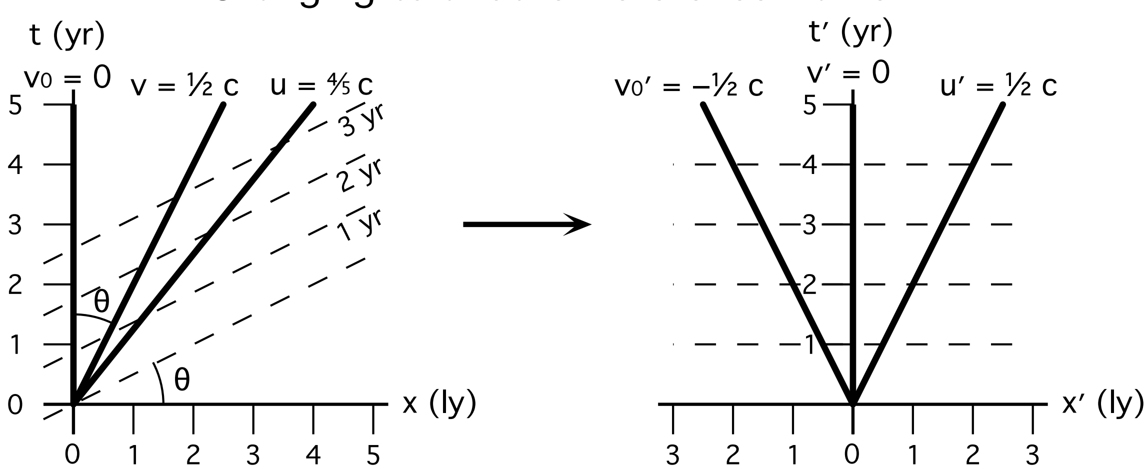
Motion in the $-x$ direction: The observer's path tilts down to the left toward 45° and her time slices tilt up, as shown at right. *(Noteworthy aside: This would be the diagram drawn by the* moving *observer in the $v = \frac{1}{2}c$ diagram on the previous page. To her, the formerly "at rest" observer is moving to the left with velocity $v = -\frac{1}{2}c$. The original observer's formerly horizontal time slices are tilted in this reference frame.)*



Multiple moving objects: The diagrams below show three observers with different velocities. In both, the time slices shown are those for the middle observer: we change from the left observer's perspective to his. After the change, the middle observer's time slices become horizontal and the velocities $\{v_0 = 0, v, u\}$ change to $\{v_0', v' = 0, u'\}$ as shown.

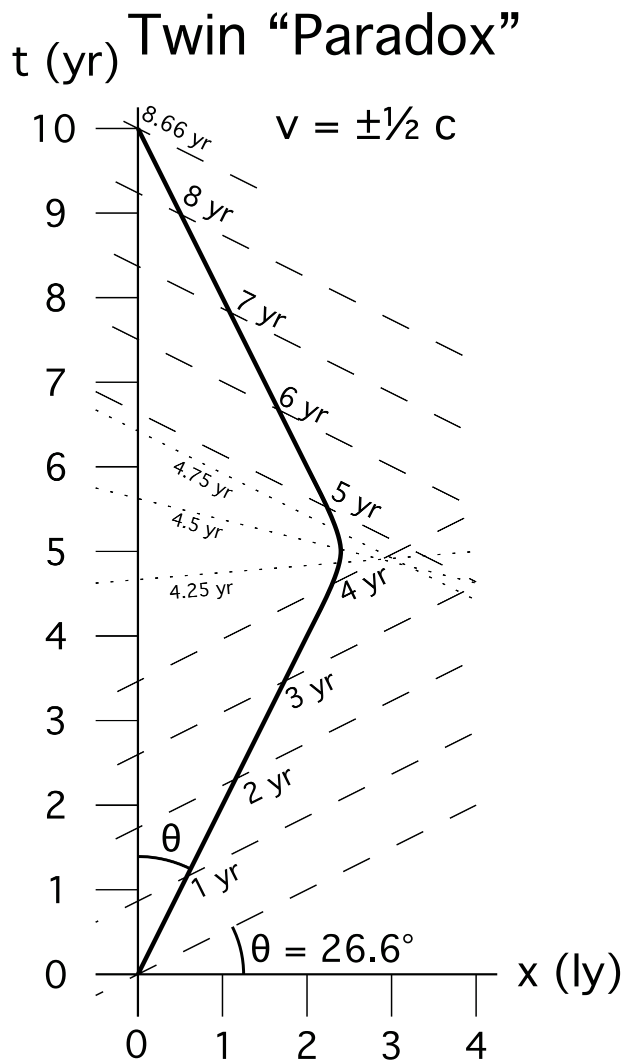
You cannot simply add and subtract velocities when changing reference frames except in a couple of special cases. Instead, use the following result (derived by measuring a moving object's position along tilted time slices using the methods above): $\boldsymbol{u' = (u - v)/(1 - \frac{uv}{c^2})}$.

Changing to another reference frame



The Twin "Paradox" in a Space-Time Diagram

Our textbook describes a 20 year old man named Speedo who takes a trip in a spaceship while his identical twin brother Goslo stays on Earth. If Speedo travels at $v = \frac{1}{2}c$ for five years (as measured on Earth) and then quickly turns around and comes home at the same rate, his path on Goslo's space-time diagram is as shown (Goslo's path is straight up the t axis). The diagram at right shows Speedo's time slices (dashed lines) at one year intervals (by his measurement). It also includes extra (dotted line) time slices at (Speedo's) quarter-year intervals between his years 4 and 5 as he turns around.



The first important observation is that when Speedo gets home, Goslo has aged 10 years (he is now 30), but Speedo has only aged 8.66 years (he is not yet even 29).

As long as Speedo moves at constant speed (away from Earth or toward it), he and Goslo each see the other as aging less quickly. (When Speedo celebrates his 24th birthday he thinks that Goslo is still only 23.46, but Goslo thinks the very same thing on his own 24th birthday.)

But that leads to the second important point: the difference between motion in an inertial frame (without acceleration) and motion that includes acceleration. During the brief period around $t = 5\,\text{yr}$ on Earth (or around $t' = 4.33\,\text{yr}$ on Speedo's spaceship) when Speedo reverses direction, the angles of his time slices change because his velocity is changing. Because those angles change while he is far away from Earth, he sees Goslo age very quickly during the brief time it takes him to turn around (by an extra 2.5 years). The faster he changes direction, the faster he sees those extra years pass back on Earth. Meanwhile, Goslo sees no such extra time pass for Speedo because Goslo's time slices never change their direction.

For this reason, we could not draw a correct space-time diagram based on Speedo as the observer "at rest." **Special relativity is only valid when the observer drawing the space-time diagram moves with constant velocity.**

As a side note, it might be possible to draw a similar kind of space-time diagram from Speedo's perspective using techniques from general *relativity, but that's beyond the scope of this class. Doing so would probably require us to draw the picture on a curved surface rather than on a flat sheet of paper: essentially, Speedo would move straight along the time axis at $x = 0$, but Goslo's path through time would have to pass over a hill, making it longer.*